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New Nondestructive Tests, from Frames

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The usefulness (Lubkin, 1976; Lubkin and Lubkin, 1979) of B-plexes, an extension of Dirac's familiar observable (Dirac, 1947), is illustrated in a computation of all nondestructive tests over a system of finite ket-space dimension n. These are tests whose B outcomes are proportional to corresponding pure final states. The unitary motions of the system and a B-dimensional register, which effect such tests, depend very simply on n vectors from a frame in B-space. Case $B > n$ is not empty, and so presents nondestructive testing with nonorthogonal final states. The step where reduction of the wave packet happens, equivalently, computation of an Everett relative state, is given in an Appendix.

1. INTRODUCTION

As nondestructiveness is prominent in von Neumann's basic exposition of quantal measurement [von Neumann (1955), reprinted in part, and with an editorial note on p. 550, in Wheeler and Zurek (1983)], I begin with von Neumann much in mind [as in Lubkin (1979*a*)].

The notion that measurement begins with a system in state x_i and an instrument in a standardized state of readiness y_0 and ends with the system still in the same state x_i but with the instrument in some state y_i signifying that the system's "i-ness" has been recorded (Wheeler and Zurek, 1983, bottom line of p. 642) is a heuristic bridge between the classical convention that a system can be observed without disturbing its state, and the situation in modern physics, where this nondisturbance is limited to the eigenstates of "the observable" of the measurement. The "nondestructiveness" that I will define and use here subjects this nondisturbance to analysis in a controlled context, and also extends to states which need not be orthogonal.

That "controlled context" is also mainly von Neumann's: The system *sys* and another entity, the instrument or its most essential representative part, which I call the register *reg*, are dealt with together as a ket in the

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tensor-product Hilbert space $H_{sys} \otimes H_{reg}$. This ket is assumed to move in time according to a linear and indeed unitary Schrödinger equation.

Von Neumann's simple schema

$$x_i y_0 \rightarrow x_i y_i \tag{1}$$

for evolution in time captures the philosophy of nondisturbance above. The x_i are the nondisturbed "eigenstates of the measurement's observable." The linearity assumed for the motion then extends (1) to the evolution of more general states

$$x = \sum_i \alpha_i x_i$$

as follows:

$$\sum_i \alpha_i x_i y_0 \rightarrow \sum_i \alpha_i x_i y_i \tag{2}$$

The output $\sum_i \alpha_i x_i y_i$ of the unitary law of motion no longer specifies a particular outcome of the measurement, but, as discussed very widely (e.g., Everett, 1957; Lubkin, 1979*a*), corresponds to *sys* winding up in state x_i and *reg* recording that by winding up in state y_i with probability $|\alpha_i|^2$. Since the original state x of *sys* was $\sum_i \alpha_i x_i$ and not any single one of the x_i (for nontrivial coefficients), we find in practice that the state of *sys* has been subjected to the disturbance of change from x to x_i if *reg* indicates the ith outcome. So as we learn the rudiments of quantum mechanics, we are thus surprised at the self-limiting quality of nondisturbing measurement: The very assumption of nondisturbance for a set of orthogonal x_i imposes disturbance rather than nondisturbance upon other states x!

Even so, it is convenient that the ith answer y_i registered on the instrument disciplines the state of *sys* to be simply x_i at the end of the experiment: it is convenient as a method for preparing states for subsequent experiments. This is of course the usual formal generalization of the way a polarizer imposes its orientation upon a photon. So, despite the disturbance of the initial state that would falsify calling the situation "nondisturbing," I yet wish to call the situation "nondestructive": The measurement told us not only the response of *reg* to some interaction with *sys*, but has also disciplined *sys*, consistent with that report, for the needs of a subsequent experiment. What is not destroyed is the value of the documentation.

That is, the conformity of "nondestructiveness" is between *reg* finally and *sys* finally; whereas the conformity of "nondisturbance" is between *sys* initially and *sys* finally.

Plan. It is my purpose to look at the unitary matrix U which effects the measurement, i.e., at the joint motion U of "*sys&reg*," to see which U actually behave nondestructively; to do so in my context of "B-plexes" for measurement (Lubkin 1974*a*, 1979*b*); and to answer some questions about nondestructiveness:[2]

Q: Is nondestructiveness just a pleasant fable which perhaps is not sustained by any actual unitary motion U?

A: No, nondestructive motions U indeed do exist.

Q: Must discipline of the final state by the outcome written on *reg* be indelibly associated with orthogonal final states?

A: Surprisingly, no. Here is where B-plexes show the way.

Q: Am I contradicting Wigner's lemma: that states which inevitably lead a certain test to different outcomes must be orthogonal in a theory where motion is unitary? (Wigner, 1952, footnote 2, p. 102.)

A: No contradiction. Wigner's states are initial; the ones here are final.

2. *B*-PLEX, FINAL STATE, ACCEPTOR, PORCUPINE, NONDESTRUCTIVE

In quantum mechanics done properly, one has states initially, but only outcomes of tests, not states, finally. Here I nevertheless explore a notion of final state.

My space H_{sys} of kets for *sys* will have n complex dimensions, n finite.

B-Plex, Acceptor (Lubkin, 1974*a*). A test with B mutually exclusive exhaustive outcomes ("bins") is represented by a list or "B-plex" a of B nonnegative $n \times n$ Hermitian matrices a_K, $K = 1, \ldots, B$, called "acceptors," with

$$\sum_K a_K = 1n \tag{3}$$

where $1n$ denotes the $n \times n$ unit matrix.

Conventions. Lower-case indices shall run over $1, \ldots, n$; Upper-case indices run over $1, \ldots, B$. Thus, $1n_{ij} = 1_{ij}$ and $1B_{CD} = 1_{CD}$ are Kronecker deltas.

An Einstein convention of omitting obvious instructions for summation will be used. But an index which appears on both sides of a relationship is by default free, not a dummy of summation, even if it appears repeatedly within a side. Explicit "Σ" is on occasion used simply for emphasis.

Porcupine. Let each a_K have rank 1; I call such a test-of-rank-1's a "porcupine" (as it has "thin quills"!). These a_K may already suggest pure

[2] B-plexes will soon be properly defined. At first, "B-plex" would merely replace an observable by the list of its eigenspaces, and would be at this level just Neumann's "resolution of identity," but then B-plexes go beyond that, to generalize the traditional Hermitian or normal observable; hence B-plexes may in this extension be called "non-Dirac observables." The definition to be given skips this history, and goes right to the quite simple extended notion.

"final states," but note that they are neither individually normalized nor mutually orthogonal. (Both conditions do hold—but only if $B = n$.[3])

Let the measurement be effected by the unitary propagation under U, of the n-dimensional system *sys* and a B-dimensional register *reg*, put together as *sys&reg*, as usual, in an nB-dimensional tensor-product ket space $H_{sys} \otimes H_{reg}$. Thus, the size of U is $nB \times nB$.[4]

Definition of Nondestructive. A "nondestructive test" must arrange to have the state of *sys* after the Kth outcome, proportional to a_K: It is this imitation of the test's acceptor by the system itself at a late time—specifically by *sys*'s evolved density matrix—that is my way here of rescuing the common notion of "final state."

3. CLAIM AND FURTHER MOTIVATION

Claim. I will show that for every porcupine, there indeed exist U's which effect it nondestructively. If $B = n$, we have a Dirac observable and the standard survival of the state in von Neumann's and in the Copenhagen viewpoint, made explicit by specifying which motions of measurement U indeed effect such survival. If $B > n$, we have an extension of this "survival of the state" to more general tests, despite the nonorthogonality of those final states.[5]

Practicality of Empirically Destructive Quantum Mechanics

It is possible to conceive of the contact between quantum mechanical theory and experiment as a mere fitting of empirical probabilities by density

[3]Proof: Case $B < n$: Empty. See the Corollary after (18). Case $B > n$: The B n-vectors $i \to v_{Ki}$ or briefly v_K representing the a_K are more numerous than the dimension n, hence they cannot be linearly independent, and *a fortiori* are not orthogonal. Case $B = n$: $\sum_K a_K = 1n$ shows each a_K to be less than $1n$, whence the vector v_K of a_K has length bounded above by 1. If any such length is actually less than 1, then, however, trace $\sum_K a_K$ falls short of $B = n$, yet also must equal trace $1n$, namely, n; from which contradiction the v_K are necessarily all unit vectors. Finally, to show that the v_K are orthogonal: Project all quantities on the orthogonal complement of one of the v's, e.g., of v_1, to get $n-1$ possibly shortened other vectors v' which correspond to null or rank-1 a'_K. The new a'_1 is zero. The $n-1$ other a'_K must therefore sum, without the help of a'_1, to projection on the $(n-1)$-space orthogonal to v_1, which has trace $n-1$. The trace argument before, in dimension n, can now be repeated in dimension $n-1$, to disallow any shortening in the projections, hence no part parallel to v_1 was actually cast away: The other v_K were indeed orthogonal to v_1; QED.

[4]For a construction in support of the convention that any unitary matrix correspond to a possible motion, see Lubkin (1974*b*).

[5]Why I use porcupines: The purpose of nondestructiveness is to have a test act as a filter, to spit out particular states as final states. Then having these states appear most sharply defined, as pure states, is a virtue. Furthermore, any test can in principle be refined to a porcupine, by adding more bins. Also rank 1 is technically convenient.

matrices for the (initial) states, and by B-plexes of acceptors for tests with B possible mutually exclusive outcomes. I call this dry outlook the "Matrix Format" (MF) (Lubkin, 1974*a*).

There is no need in MF for any notion of final state: You begin with a state, but end with an outcome of a test, not with a state. If then you wish to do another experiment, you make a brand new state. How you make states, how you perform tests on them, is part of everyday life, and is not necessarily comprehended within a quantal model.

It may be clarifying indeed to set forth MF, albeit tersely:

MF. The probability $p(I, J, K)$ that state I followed by test J yields outcome K is found directly, if laboriously, by making many trials of preparing state I, then applying test J, and cumulatively recording which bin K catches the outcome.[6] Comparison to "theory" is by cumbersomely adjusting the Ith state matrix ρ_I and the Jth test's B-plex $(a_{J1} \ldots a_{JB})$ of acceptor matrices a_{JK} (whose typical qrth matrix element would be a_{JKqr}) so as to best fit

$$p(I, J, K) = \text{trace } \rho_I \, a_{JK} \tag{4}$$

If the size $n \times n$ of all these matrices is kept small, there may easily be enough data $\{p(I, J, K)\}$ to freeze them—all the ρ's and all the a's—up to an overall unitary conjugation.

Both the states I and the tests J are called into being by ordinary human effort. So quantum mechanics is, as it were, immersed in a sea of everyday affairs. No quantally produced final state is called for to initiate a new trial of an experiment; rather, one follows some blueprints to prepare "state I," etc. This is in general conformity with Bohr's dictum that quantal experiments are initiated and terminated "classically," except that not even classical mechanics need play a part in the laboratory, a clarification offered by Schrödinger in 1935 (Moore, 1989, p. 313) and by Weyl (1949, Appendix C).

This insularity is well known, and it is convenient for carrying out particular projects: each project has its own states, tests, and experiments. *There need be no universe* to cut up into those separate experiments.

Is quantum mechanics then incomplete in its need for this sustaining world of everyday affairs? Is the language of Dirac's kets and bras (or acceptors) as dependent perhaps *in principle* upon the language of ordinary speech and experience? Or are quantal "explanations" for numerous details

[6]The rule that capitalized indices run over $1, \ldots, B$ is evidently inapplicable to I and J here. Also, the number of bins possessed by test J may vary from test to test, hence B here should really be J-tagged: BJ instead of B; this is not done, to avoid clutter. Similarly, "state I" abbreviates "the state bearing number or tag I in a catalogue" (e.g., Giles, 1976).

of ordinary experience a sufficient web for us to be justified in feeling that we nevertheless live in a universe that is some Laplacian though quantal machine?

Need Nevertheless for an Ongoing State

Perhaps a notion of final state, that is, a(n initial) state for "the next experiment," related in some simple way to the outcome of an (earlier) experiment, while superfluous in the conciseness of MF, may yet guide us to a scientific grasp of this shell of everyday affairs within which the quantal experiments seem to be housed. Perhaps "final state" can help to link quantum mechanics to an ongoing "I-time" (Einstein, 1953, esp. p. 3) in dealing with reality and consciousness.

That is, how do Bohr's individual experiments link up into some sort of world?

I should note that von Neumann (1955) presents as the first of his "two fundamentally different types of interventions" (Wheeler and Zurek, 1983, p. 553) an alteration of his (initial) density matrix conceived of as representing an ensemble of physical systems, due to a quantal measurement associated with a Hermitian observable. Von Neumann's ensemble after measurement throws together all instances of outcomes; he does not sort out subensembles according to the outcome K. Hence, von Neumann's ensemble after measurement, strangely, represents the result of a measurement relative to an observer who has not yet learned the outcome (not so strange: a deft avoidance by Neumann of "reduction of the wave packet")—and for the story of how the entropy attending this evident lack of information yet survives as entropy of measurement even after the outcome is known, see Lubkin (1987): the loss of information comes about when the previous contents of the register on which the answer is written are erased. I have since learned that Landauer (1961) and others (Leff and Rex, 1990) have dealt extensively with the entropy of erasure. As I may otherwise seem to have forgotten erasure here, let me note that the erasure took place early, when *reg* was preset to state e_1 (see below).

Von Neumann (1955) thus features a final state prominently, while nevertheless leaving much unsaid, so that I have felt it best to make a new business of it here.

4. LINK BETWEEN ACCEPTORS AND THE MOTION

a_{Kjm} for a General Acceptor, in Terms of U: An Old "onto" Theorem. Aside on B-plex As "Microscope"

In Lubkin (1974*a*,*b*), I showed that to any B-plectic list of matrices a there corresponds an actual test with B outcomes whose B-plex is a. As the paradigm of that paper is also the workhorse here, it will be well to review it; indeed, the model here is a slight extension of that.

The system starts in state ρ, and the register starts in its preset state e_1, the first[7] of a list e_K, $K = 1, \ldots, B$, of projections on an orthonormal basis $|e_K)$, $K = 1, \ldots, B$, of B-dimensional H_{reg}. Thus, $e_K = |e_K)(e_K|$.

Then, general motion U ensues.

Finally, *reg* is subjected to a traditional and nondegenerate Dirac observable e', whose B mutually orthogonal eigenprojections are the e'_K, $K = 1, \ldots, B$, with eigenkets $|e'_K)$. Thus, $e'_K = |e'_K)(e'_K|$. The expansion

$$|e'_K) = c_{KL}|e_L) \tag{5}$$

defines a unitary matrix c of coefficients c_{KL} relating the two bases. In Lubkin (1974*a*), all B-plexes were already found without the need for such a rotation c, but as I am here aiming at the most general $nB \times nB$ motion U consistent with nondestructiveness, I must now avoid that specialization.

In the context of the larger space $H_{sys} \otimes H_{reg}$, the resolution of the final observable is instead in terms of the fattened projections $E'_K = 1n \otimes e'_K$.

Thus, the probability of outcome K is

$$\text{Trace } U\rho \otimes e_1 U^{\text{adj}} E'_K \tag{6}$$

Capitalized "Trace" here refers to all labels, i.e., it is of an $nB \times nB$ matrix. To determine acceptors a_K for *sys* itself, this probability is equated to

$$\text{trace } \rho\, a_K \tag{7}$$

which is a computation with smaller, $n \times n$ matrices only. In components, and with

$$e_{KCD} = 1_{CK} 1_{KD} \tag{8}$$

diagonal, the equation of (6) and (7) determines the acceptors a_K:

$$c^{\text{conj}}_{KA} U_{iAj1} U^{\text{conj}}_{iDm1} c_{KD} = a_{Kmj} \tag{9}$$

Equation (9), in the simplified version where c was a unit matrix, was seen in Lubkin (1974*a*) to map all the unitaries U onto all the B-plexes a. This surjectivity established that all B-plexes indeed occur as quantal tests.

It may be well to remark here on the virtue of the B-plex as a "microscope": The original test, in the grammar of the larger nB-space of *sys&reg*, had an unexciting B-plex $E' = (E'_1 \ldots E'_B)$ of mutually orthogonal projections. It is by reducing this grammar down to *sys* alone that the

[7]Register's ket e_1 plays the part of y_0 in the Introduction.

general B-plex $a = (a_1 \ldots a_B)$ comes into view. Were B-plexes limited by fiat to lists of projections, this "focusing down from *sys&reg* to *sys* itself" would be interdicted.

a_{Kjm} for a Porcupine As a Condition on U: Imposing Rank 1

Equation (9) will now be used to determine how U should be restricted so as to produce a porcupine for a: Note that (9) is of the form

$$\sum_i u^{\mathrm{adj}}_{Kmi} u_{Kij} = a_{Kmj} \tag{10}$$

where

$$u_{Kij} = c^{\mathrm{conj}}_{KD} U_{iDj1} \tag{11}$$

Note also the more complete version,

$$u_{iKjL} = c^{\mathrm{conj}}_{KD} U_{iDjL} \tag{12}$$

in which form it may be said that the u_{iKjL} constitute an $nB \times nB$ unitary matrix, which follows directly from the unitarity of U and of c.

Fix K.

Each (10) term for fixed i in the sum over dummy i is a nonnegative multiple of a 1-dimensional projection. Hence, the sum over i would surpass rank 1 (the defining condition of "porcupine"), unless the terms in the sum over i are proportional. The ith term projects (mod normalization) on an ith vector whose jth component is u_{Kij}. So these several vectors must be complex-proportional, i.e., the ijth u-element is some ith complex multiplier $r(\mathrm{ow})_i$ times some single vector's jth component, $c(\mathrm{olumn})_j$: thus, not forgetting the bystanding fixed index K, we have

$$u_{Kij} = r(\mathrm{ow})_{Ki} c(\mathrm{olumn})_{Kj} \tag{13}$$

with of course no sum on K, and indeed with i, j, K all free.[8]

As (14), below, is (by inspection) already of form (13), further comment on (13) would be moot.

5. NONDESTRUCTIVENESS

$E'P'E'$ Conveyance of Nondestructiveness for a Porcupine to U

I now must do a reduction of the wave packet, or pass to an Everett relative state (Everett, 1957), in order to capture algebraically the knowledge

[8]Equivalently, each $n \times n$ matrix $(i, j) \to u_{Kij}$ is of rank 1.

that the outcome is the Kth, and it is here that I could easily be begging the question of nondestructiveness. So I will for clarity immediately display the immediate result (14), then the decisive $E'_K P' E'_K$ step, then a clearer "answer" in terms of vectors v, and shunt justification of $E'_K P' E'_K$ to the Appendix.

Theorem:

$$u_{Kij} = \lambda_K z_{Ki} z_{Kj}^{\text{conj}} \tag{14}$$

expresses nondestructiveness in terms of B unit n-vectors z_{Ki} and B complex λ_K. Conditions are given after the following derivation.

Proof. The final (unnormalized) state of *sys&reg*, after the Kth outcome is known, is taken to be $E'_K P' E'_K$, where P' is the state UPU^{adj} after the U-motion: for this collapse of the state, see the Appendix. Then the final state of *sys* itself is proportional to the Landau trace or partial trace of this, over the *reg* space's indices. Nondestructiveness, by definition, demands that this final state of *sys* itself be proportional to the Kth acceptor a_K; thus, $\text{LandauTrace}(1n \otimes e'_K U\rho \otimes e_1 U^{\text{adj}}\, 1n \otimes e'_K) \propto a_K$ expresses nondestructiveness. In components and through (9), this is

$$\begin{aligned} &1_{mr} e'_{KCD} U_{rDsF} \rho_{st} e_{1FG} U^{\text{conj}}_{wItG} 1_{wj} e'_{KIC} \\ &\propto c^{\text{conj}}_{KJ} U_{iJj1} U^{\text{conj}}_{iHm1} c_{KH} \end{aligned}$$

which distills to

$$u_{Kms} \rho_{st} u^{\text{conj}}_{Kjt} \propto u_{Kij} u^{\text{conj}}_{Kim}$$

to be true for all states ρ. Temporarily drop the spectator label K for clarity, and note that we then have the $n \times n$ matrix relation

$$u\rho u^{\text{adj}} \propto u^{\text{adj}} u \qquad \text{for all densities } \rho \tag{15}$$

Insert the (normalized) unit matrix for ρ into (15), then trace, to see[9] that

$$uu^{\text{adj}} = u^{\text{adj}} u$$

This commutativity with the adjoint characterizes a normal matrix: hence u is normal. So the eigenspaces of u are orthogonal. Accordingly, unitarily diagonalize u.

Next, insert an off-diagonal ρ into (15), and see that the left side, $u\rho u^{\text{adj}}$, therefore also has an off-diagonal part connecting any two nonzero diagonal-u places. Yet the right side is diagonal. This discrepancy establishes that all but one element of diagonalized u must vanish. So, diagonalized u

[9] The alternative $u = 0$ is excluded by my demand that each quill a_K have rank 1.

is some nonzero complex multiple of projection on some unit n-vector z.[10]

This gives the above formula, (14), for the original, prediagonalized u, when we restore the suppressed index K. QED

Conditions on λ_K and on z_{Ki} follow equivalently from unitarity of U or from (3):

Substitution of (14) into (10) gives

$$\sum_i z_{Ki} z_{Kj}^{\text{conj}} |\lambda_K|^2 z_{Ki}^{\text{conj}} z_{Km} = a_{Kmj}$$

which simplifies, because each z_{Ki} for fixed K is a unit n-vector, to

$$z_{Kj}^{\text{conj}} |\lambda_K|^2 z_{Km} = a_{Kmj}$$

It is now best to absorb the $|\lambda_K|$:

$$|\lambda_K| z_{Kj} = v_{Kj} \tag{16}$$

Then

$$v_{Kj}^{\text{conj}} v_{Km} = a_{Kmj} \tag{17}$$

and summing over K yields the unit matrix $1n$, from (3) in the definition of "B-plex"; thus,

$$\sum_K v_{Kj}^{\text{conj}} v_{Km} = 1_{mj} \tag{18}$$

which is to say that the v_{Kj} are n orthonormal B-vectors.

The conditions packed in the orthonormality of v will presently be seen to be just enough to produce an answer free of any further condition.

Conditions. Thus, the conditions are simply that the v_{Kj} constitute n orthonormal B-vectors, or, more picturesquely, that they are the "first" n vectors of a B-frame. ("Frame" is meant to signify orthonormality.)

Corollary. Case $B < n$ is empty.

Brief Detour on the Corollary. Indeed, more than B orthonormal vectors in B-space is impossible; QED. Hence, our nondestructive topic does not exist for theorists confined to "questions," that is, to tests with only $B = 2$ outcomes, except for the subcase $B = n = 2$.

[10]The λ_K factors mask any effect on the *answer* for u that normalization of the n-vectors z might have had, hence the renormalization [in (16)] of z to v causes no problem. The z normalization that thus gets "lost" is used only to shape the following argument.

(14) will be called (19) when rewritten to eliminate z in favor of v; this is listed below, within an exhaustive statement of the "answer," that is, a list of conditions both necessary and sufficient to define a motion U which effects nondestructive measurement.

The Answer, in Terms of a Schmidt Process in B-Space. How This is a New "onto" Theorem

Thus,

$$a_{Kmj} = v_{Kj}^{\text{conj}} v_{Km} \tag{17}$$

and

$$u_{Kij} = \theta_K \text{ norm}_K\, v_{Ki}/\text{norm}_K\, v_{Kj}^{\text{conj}}/\text{norm}_K \tag{19}$$

which with the aid of (11) and the unitarity of c gives

$$U_{iDj1} = c_{KD}\theta_K \text{ norm}_K\, v_{Ki} v_{Kj}^{\text{conj}} \tag{20}$$

where

$$\text{norm}_K = \left(\sum_p |v_{Kp}|^2\right)^{1/2}$$

and where

$$v_{Kp} \text{ is } n \text{ orthonormal } B\text{-vectors} \tag{18}$$

shows how to build the 1 sector of U out of n arbitrary orthonormal B-vectors v, out of B arbitrary phase factors $\theta_K = \exp(i\phi_K)$, and out of the arbitrary unitary twist or second B-frame c.

Proof. Formula (17) follows from (19) and (10). Formula (19) is merely (14), transcribed to favor v. Hence, the conditions cited are indeed necessary.

The derivative nature of (17) also refers the question of sufficiency entirely to the question of the consistency of (20). Specifically, that question is whether U so specified from some B-frame v freely given (and c and θ_K) is unitary—hence is to be regarded a possible motion (see footnote 4).

As the n columns numbered $j1$ of U as given by (20) are identically orthonormal, they are easily Schmidt-completed to a full U. Hence, the U's given by (20) do indeed extend to full unitary U's; QED necessity and sufficiency for the v a freely given B-frame, the ϕ free phases, and a free twist or other B-frame c, to define a motion U.

So the U's cited do unitarily implement, as nondestructive tests, all those porcupines whose a_K are given by the $v^{\text{conj}}v$ formula (17). Having done with "necessary" and "sufficient," what is yet left? I wish to show that, as v runs over all possible B-frames, the a's hit by (17) are *all* of the porcupines; that (17) as a mapping is onto.

Proof That (17) *Is Onto.* The form $v^{\text{conj}}v$ for any one a_K says only that that a_K has rank 1, which merely affirms that the porcupine property has been built in. The only remaining doubt stems from possibly restrictive interrelations between the a_K for distinct K which might follow from our only condition on the vectors v, namely, the B-vector orthonormality

$$\sum_K v_{Kj}^{\text{conj}} v_{Km} = 1_{jm} \tag{18}$$

but (18) merely demands [through (17)] that

$$\sum_K a_{Kmj} = 1_{jm} \tag{3}$$

which correlation between the a_K is part of defining "B-plex." Hence, (18) is B-plectically identically met; QED onto.

6. CLOSE RONDO: BACK TO DISTURBANCE AND BACK TO VON NEUMANN'S SCHEMA

When are our nondestructive tests also nondisturbing?

Case $B = n$

The standard picture of the textbooks is borne out: If initial state ρ is itself a_K, then trace ρa_K, the probability of the Kth outcome, is 1. These ρ are of course the eigenstates of any nondegenerate Dirac observable A whose eigenprojections are the a_K; hence it is precisely eigenstates of A that are not disturbed. (Disturbance of the other states x was reviewed in the Introduction.) Repeated observation with such a test first casts any mixed state ρ into a state a_K, then subsequently reports K-ness without further disturbance.

The General Case, $B \geq n$

Here arbitrary initial ρ is also cast into state $a_K/\text{trace}\ a_K$ whose ket (mod phase) is $z_K = v_K/|v_K|$, with probability

$$\begin{aligned}
\text{trace}\ \rho a_K &= \text{trace}\ \rho v_K v_K^{\text{adj}} \\
&= \text{trace}\ \rho z_K z_K^{\text{adj}} |v_K|^2 \\
&= \text{trace}\ \rho z_K z_K^{\text{adj}}\ \text{trace}\ a_K
\end{aligned}$$

But now even one of the z_K given initially has nonvanishing probability of being disturbed: The chance of z_K "scattering" to z_L is

$$\begin{aligned}\text{trace } \rho a_L &= \text{trace}[(a_K/\text{trace } a_K)a_L]\\ &= \text{trace } z_K z_K^{\text{adj}} v_L v_L^{\text{adj}}\\ &= \text{trace } z_K z_K^{\text{adj}} z_L z_L^{\text{adj}} |v_L|^2\\ &= |\langle z_K | z_L\rangle|^2 |v_L|^2\end{aligned}$$

In particular, the chance that z_K is undisturbed (mod phase) by such a measurement is only $|v_K|^2 = \text{trace } a_K$, less than 1, if the Kth acceptor a_K is short. Such instability to scattering is of course in conformity with Wigner's lemma.[11]

If on the other hand a_K is of full trace 1, that a_K is indeed orthogonal to all other a_L's, and is entirely apart from the crowding implicit in having more than n outcomes; this subcase can be separated out by first splitting our B-plex into sharp and 1-free parts (Lubkin, 1974*a*, 1979*b*).

Overket

We now apply our general nondestructive U to a general *pure* initial state of the sort contemplated in (2), namely, to $|x\rangle|e_1)$, and find that the necessarily pure final ket obtained from unitary motion is, for the nondestructive motions U, indeed a reasonable extension of Neumann's (2).

In components on the basis $|b_i\rangle$ for *sys* and the basis $|e_K)$ for *reg*,

$$|x\rangle|e_1) = x_{iA}|b_i\rangle|e_A)$$

where $x_{iA} = x_i 1_{A1}$. The final *sys&reg* ket or "overket" is then, in components,

$$x'_{iD} = U_{iDjA}x_{jA} = U_{iDj1}x_j$$

Then (20) gives

$$x'_{iD} = c_{KD}\theta_K \text{ norm}_K^{-1} v_{Ki}v_{Kj}^{\text{conj}}x_j$$

or

$$x'_{iD} = c_{KD}\theta_K \text{ norm}_K z_{Ki}z_{Kj}^{\text{conj}}x_j$$

The analogy to (2) becomes clearer if the components are wrapped up again into whole kets. I use both (end brac)ket symbols to mark a *sys&reg*

[11] Wigner's lemma is stated in the last Q of the Introduction.

ket, thus, |ketname⟩⟩); in this convention, we have

$$|x'\rangle\rangle = x'_{iD}|b_i\rangle|e_D) = c_{KD}|e_D)\theta_K \text{ norm}_K\, z_{Ki}|b_i\rangle z^{\text{conj}}_{Kj} x_j$$

where all indices are dummies, or

$$|x'\rangle\rangle = \sum_K \theta_K \text{ norm}_K \langle z_K|x\rangle|z_K\rangle|e'_K)$$

$$= \sum_K \theta_K \text{ norm}_K^{-1}|v_K\rangle\langle v_K|x\rangle|e'_K)$$

$$= \sum_K \theta_K\langle z_K|x\rangle|v_K\rangle|e'_K) \tag{21}$$

where the notation $|e'_K)$ for $c_{KD}|e_D)$ recalls (5), and where the B unit n-vectors $z_{Ki}|b_i\rangle$ are called $|z_K\rangle$.

In the traditional situation $B=n$, the norm_K are all 1, and only the phases θ_K distinguish (21) from the final overket of (2); they signify the most general U imposing an extra and obvious final rephasing. The θ_K cannot be absorbed into the z_K or the v_K in the first two versions of (21), as bra and ket rephasings cancel, but perhaps confusingly could have been absorbed in the third version, but only by breaking the positivity of my renormalizing in (16).

When $B>n$, the results on transitions for repeated measurements already developed using traces are easily recovered from inner products with (21). The terms in the sum on K are orthogonal even though the $|z_K\rangle$ are not, because the $|e'_K)$ are.

The probability p_{LK} of an $L\to K$ transition is $(\text{norm}_K \cos_{KL})^2$, where $\cos_{KL} = |\langle z_K|z_L\rangle| = \cos_{LK}$ is K-L symmetric, but since usually norm_K differs from norm_L (case $B>n$), p_{LK} usually differs from p_{KL}. This is not surprising, as $L\to K$ is, more completely,

$$|z_L, e_1\rangle\rangle \to |z_K, e'_K\rangle\rangle$$

and so is not simply related to the $K\to L$ transition,

$$|z_K, e_1\rangle\rangle \to |z_L, e'_L\rangle\rangle$$

In a sequence of similar measurements, we do not accept the final *sys&reg* ket $|x'\rangle\rangle$ unchanged in going to the next measurement, but first reset *reg* to e_1. What happens when U hits an unreset ket $|z_L, e'_M\rangle\rangle$, for M not 1, is not interesting, as the elements of U that come up are those in the arbitrary Schmidt completion, outside the scope of (14) and even of (13); so to consider motion reversal would be pointless. Yet, in the case $B=n$, we do have $p_{LK}=p_{KL}$, in spite of the dissociation from motion reversal.

7. CODA: THERMAL INITIALIZATION OF THE REGISTER

What about having *reg* start in a mixed state?

If a nondestructive final state is a pure *sys&reg* ket, then there must be a garbage can (gc) aside from *sys&reg* to accept the entropy. Discussing that would violate the minimal policy here of measurement only through unitary motion of *sys&reg*. Yet the strict correlation between *sys* and *reg* finally can admit randomized phases *between* the outcome-indexed terms, hence can have entropy up to von Neumann's entropy of measurement

$$vN = -\sum p_K \ln p_K$$

Hence, that much entropy could perhaps be channeled internally.

Indeed, vN was the entropy of the noisy initial state used in Lubkin (1987) as solution to the problem of generating the least entropy of erasure in the writing of the measurement's answer on a register.

Nevertheless, in spite of the suggestiveness of this matching of entropies vN, it turns out to be impossible to devise a unitary motion that will work nondestructively for an entropic initial state of *reg*:

No-Go Theorem. The initial state y_0 of *reg* must be pure for a test to work nondestructively.

Proof. Suppose contrarily an impure y_0 and a motion U of *sys&reg* leading to a nondestructive test with porcupine a whose Kth acceptor a_K is $v_K v_K^{\mathrm{adj}}$. Diagonalize y_0, and let its constituent projections be the e_L, $L = 1, 2, \ldots$; there are at least two terms, else y_0 would have been pure.

The effect of the U motion on the candidate ensemble y_0 can be obtained by convexly combining its effect on starts with those separate e_L. If for any such e_L, the outcome K gave a final *sys* state not proportional to a_K, then the convex combination would also not be proportional to a_K, violating nondestructiveness. Thus, each contributing e_L induces a nondestructive test with the Kth acceptor a_{KL} proportional to a_K. Hence, we have (14) edited with L instead of 1, for each contributing e_L in y_0, and (19) becomes

$$u_{iKjL} = \theta_{KL} \operatorname{norm}_{KL}^{-1} v_{KLi} v_{KLj}^{\mathrm{conj}}$$

where, however, the v_{KL} dependence on L deals at most with the vector's length.

Fixing j, L here selects a column from the components of the unitary matrix u, u_{iKjL} [see (12)]. The columns jL for fixed j and each contributing L are complex-proportional. The columns are also unit nB-vectors, from unitarity. Dot products between proportional unit vectors have absolute value 1. Unitarity of u, however, demands zeros for products between distinct contributing L values, of which there are at least 2. That is intolerable. Hence, no nondestructive motion U exists for the mixed start y_0 of any reg. QED.

Hence, a thermally quenched *reg* bars nondestructiveness—with no extra device. My use (Lubkin, 1987) of thermal prequenching for minimal loss of information in overwriting was indeed for a measurement assumed to be so strongly disciplining as to leave no entropy after the outcome is known, which resembles the present notion of nondestructiveness, but is in principle more general; *sys* could for example be annihilated, leaving only its mark on *reg*. Or one could use the present scheme after first forcing *reg* to e_1 before all else; dethermalizing it! But that is not entropically optimal. It is doubtful, then, that the thermally quenched start can be made to work with mean entropic production limited to vN each trial. I cannot claim this "disproved," since argument here is limited to mechanisms within *sys&reg*'s nB Hilbert dimensions.

8. SIGNIFICANCE OF THE NONDESTRUCTIVE PRODUCTION OF STATES AS HEREIN DESCRIBED

But Everybody Already Knows How to Produce Nonorthogonal States!

Everybody already knows how to produce nonorthogonal states for something like MF, by a variety of arbitrary state-producing arrangements, e.g., by using nonperpendicular Polaroids on photons. Hence, the present result lies not in the production of orthogonal or nonorthogonal states in itself, but in the minimality of the tools used: Both nonorthogonal preparation with $B > n$ and orthogonal preparation with $B = n$ tack onto *sys* only the one B-dimensional entity, *reg*. That is minimal, because a dial with B Wigner-stable settings does need B mutually orthogonal settings (see footnote 11).

If anything that can be engineered joins physics, then the ability to model any consistent pattern on a computer would incorporate anything noncontradictory; "anything goes" (save for possible finitary restriction). To block this, "physics" must be the *art* of telling "natural" apart from "artificial"! In this admittedly feeble light, the minimality of *reg* earns it the "natural" sticker. But is the construction in Lubkin (1974*b*), my support for all U's being possible motions, then, too broadly grammatical?

APPENDIX

Here we consider the collapse of the final state P' of *sys&reg* to the normalized Landau trace of $E'_K P' E'_K$ as the state of *sys* after the outcome read off *reg* is known to be the Kth, or relative the Kth outcome.

A1. Preliminary Algebra

Let the final state at first be uncollapsed. That is just $U\rho \otimes e_1 U^{\text{adj}}$ here, but the point may as well be couched more generally, so call the uncollapsed state simply P'.

I shall read the collapsed state of *sys* from the expectation value of the Dirac observable h on *sys*, while the outcome of the e' test on *reg* is known to be the Kth.

So the state is P'_{iCjD}, and test is some cousin of $H_K = h \otimes e'_K$, in components, of $H_{KiCjD} = h_{ij} e'_{KCD}$. The Dirac *sys&reg* observable H_K has the expectation value

$$\begin{aligned}\text{Trace } P'\, H_K &= P'_{iCjD} H_{KjDiC}\\ &= P'_{iCjD} h_{ji} e'_{KDC}\\ &= \rho_{Kij} h_{ji}\\ &= \text{trace } \rho_K h\end{aligned}$$

where

$$\rho_{Kij} = P'_{iCjD} e'_{KDC} \tag{22}$$

Lack of involvement with the initial state allows us in this Appendix to diagonalize the final e'_K basis. Accordingly, so as not to err when the e and e' both enter, correct portage of these results to the main text uses a basis-free notation. But here, we will also use

$$e'_{KDC} = 1_{DK} 1_{KC}$$

which gives the more compact version

$$\rho_{Kij} = P'_{iKjK} \tag{22}$$

with no sum on K. This ρ_K, however, fails to be a density matrix, because its trace is

$$t_K = \rho_{Kjj} = P'_{iKiK} \tag{23}$$

summed on i but not on K, whereas K must also be summed on to give a state's normalization, 1. The quantities just introduced through this unfocused algebra will nevertheless be useful below.

A2. Ensembles

Full, uncollapsed P' provides a useful ensemble, in the style of von Neumann, on which we effect simultaneous measurement of the two commuting Dirac observables $H = h \otimes 1B$ and $E'_K = 1n \otimes e'_K$.

Protocol 1. The answer for E'_K is 1 or 0. Throw away the H answer whenever E'_K comes out 0. But when E'_K comes out 1, do log the answer for H.

This on the face of it represents "the expectation of h when the outcome of e' is known to be the Kth," hence this protocol 1 is what is to be captured in formula (26) below.

But consider first the more easily captured following protocol:

Protocol 2. Record the H value when the E'_K value is 1 as before, but now record 0 when the E'_K value is 0.

Then we do get as expectation value the *sys&reg* expectation of H_K worked out in the preliminary algebra,

$$\text{expectation2} = \text{Trace } P' H_K = \text{trace } \rho_K h \tag{24}$$

since $H_K = HE'_K$, the ordinary product of H and E'_K, indeed does come out as the H or h value when E'_K is 1, and 0 when E'_K is 0.

This expectation2 does not do for the desired protocol 1, because while the throwing away of zero answers for H_K when E'_K is 0 do not affect the accumulation in the numerator, it does reduce the size-of-sample denominator in an empirical determination of an expectation value.[12] Indeed,

$$\text{expectation1} = \text{expectation2} * (\text{total trials}/\text{allowed trials})$$

The count of allowed trials as compared to total trials is the expectation value of E'_K itself. That is Trace $P' E'_K = t_K$ [see (23)]. Hence,

$$\begin{aligned} \text{expectation1} &= \text{Trace } P' H_K * (1/\text{Trace } P' E'_K) \\ &= \text{Trace } P' H_K / t_K \end{aligned}$$

or

$$\text{expectation1} = \text{trace } \rho'_K h \tag{25}$$

where

$$\rho'_K = \rho_K / t_K \tag{26}$$

is now properly normalized, to trace 1. Equations (26), (22), and (23) give us the state ρ'_K for *sys* as an ensemble relative to knowledge that *reg* has outcome K, because it is ρ'_K which is featured in the correct expectation (25).

[12]The application of this Appendix is to proportions culminating in (15), not to an equation. Hence, normalization is moot, and it may be objected that that is what is being examined. Discussion is nevertheless justified in order to allay doubts about a suspiciously unnormalized state $E'_K P' E'_K$, as well as to round out "reduction of the wave packet."

Again, the final state is ρ'_K, where

$$\rho'_{Kij} = P'_{iKjK}/P'_{aKaK}$$
$$= \text{Landau trace of } E'_K P' E'_K \quad \text{normalized} \tag{26}$$

But this should also reduce to Everett's (1957) prescription for a *relative state*, namely, to a partial inner product, if we go back from density matrices to kets.

The encompassing state to be treated is P'. The "observer" *reg* relative to whom the partial inner product is to be taken is in the state e'_K, which is pure. Were P' also pure, so that kets for both P' and e'_K existed—let them be X_{iA} and Y_A—the reduced ket x in Everett's prescription would be the partial inner product of X with Y:

$$x_i = X_{iA} Y_A^{\text{conj}} \tag{27}$$

giving as reduced density matrix

$$\rho_{Kij} = x_i x_j^{\text{conj}} = X_{iA} Y_A^{\text{conj}} X_{jC}^{\text{conj}} Y_C = P'_{iAjC} e'_{KCA}$$

which is the same as the unnormalized identically named expression for protocol 2 above, and is of course to be upgraded to ρ'_{Kij}, as above, for protocol 1. (Indices A and C are dummies.)

This argument has treated P' as pure, so as to admit a representative vector X in ket space for being quite literal about "partial inner product," but we already have the same expressions ρ_{Kij} for protocol 2 and ρ'_{Kij} for protocol 1, free of restriction of state P' to purity. Thus, the discussion of $E'_K P' E'_K / t_K$ above, couched in Neumann's language of ensembles, does extend Everett's relative-state partial inner product beyond purity of the enveloping state P'.[13]

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[13]In my application, $P' = UPU^{\text{adj}} = U\rho \otimes e_1 U^{\text{adj}}$ is pure precisely when ρ is pure.

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